# ACCELERATION OF A CONDUCTING FLUID IN MAGNETOHYDRODYNAMIC CHANNELS WITH CONSIDERATION OF THE SELF AND MUTUAL INDUCTION IN THE EXTERNAL CIRCUIT

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A solution is presented for the problem of accelerating a viscous incompressible conducting fluid in a magnetohydrodynamic channel whose electrodes close an external circuit exhibiting resistance and inductance, and connected by mutual induction to a secondary circuit. The flow regime in the channel is assumed to be laminar.

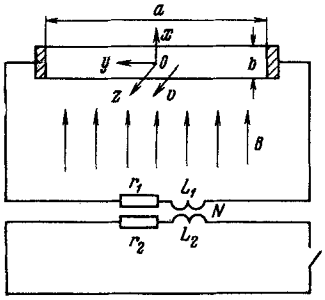


Fig. 1

Let us consider the unsteady flow of a conducting fluid in a long channel of rectangular cross section in the presence of a uniform magnetic field $B_x = B_0$ (Fig. 1). We will assume the two channel walls $x = \pm b/2$ to be thin and made of metal of low conductivity, and we will assume the other two walls $y = \pm a/2$ to be electrodes of good conducting material and closing an external circuit with resistance $r_1$ and inductance $L_1$. Let this circuit be connected by mutual inductance N to a secondary circuit with resistance $r_2$ and inductance $L_2$. We also assume that $a \gg b$, so that the change in the flow parameters along the y axis in comparison with the change along the x-axis can be neglected. Under these conditions, the system of magnetohydrodynamics equations takes the form

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta \frac{\partial^2 v_z}{\partial x^2} - j_y B_0,$$
$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t},$$
$$j_y = -\frac{1}{\mu}\frac{\partial B_z}{\partial x} = \sigma (E_y + v_z B_0). \quad (1)$$

Here v is the fluid flow velocity, p is the pressure, j is the electric current density, E is the electric field strength; $\rho, \eta, \sigma$, and $\mu$ are the density, dynamic viscosity, conductivity, and magnetic permeability, respectively, of the fluid.

In the cases of practical interest it may be assumed that $E_y$ does not depend on x, and is a function only of time [1]. Then, in accordance with Kirchhoff's second law, we may write the following system of equations for the total current flowing in the mutually coupled circuits

$$r_1 I_1 + L_1 \frac{dI_1}{dt} + N\frac{dI_2}{dt} + E_y a = 0,$$
$$r_2 I_2 + L_2 \frac{dI_2}{dt} + N \frac{dI_1}{dt} = 0. \quad (2)$$

When we consider (2), Eq. (1) reduces to the following system:

$$\frac{\partial v}{\partial t} = P(t) + \nu \frac{\partial^2 v}{\partial x^2} - Mr\frac{\nu}{ab}\sqrt{\frac{\sigma}{\eta}} I_1 + \frac{M^2 \nu}{b^2}(U - v), \quad (3)$$

$$(r_1 + r) I_1 + L_1 \frac{dI_1}{dt} + N\frac{dI_2}{dt} = UB_0 a, \quad (4)$$

$$r_2 I_2 + L_2 \frac{dI_2}{dt} + N\frac{dI_1}{dt} = 0,$$

$$U = \frac{1}{b}\int_{-1/2 b}^{1/2 b} v\, dx, \quad M = bB_0 \sqrt{\sigma/\eta},$$
$$r = \frac{a}{\sigma b l}, \quad P(t) = -\frac{1}{\rho}\frac{dp}{dz}, \quad \nu = \frac{\eta}{\rho}. \quad (5)$$

Here U is the mean fluid flow velocity, M is the Hartmann number, and r is the internal resistance of a channel of length $l$.

The boundary conditions for (3) will be

$$v = 0, \qquad x = \pm 1/2 b. \quad (6)$$

The initial conditions may be assigned in the form

$$v(x, 0) = I_1 = I_2 = \frac{dI_1}{dt} = \frac{dI_2}{dt} = 0,$$
$$\frac{d^2 I_1}{dt^2} = \frac{P_0 a B_0 L_2}{L_1 L_2 - N^2} \quad \text{at} \quad t = 0. \quad (7)$$

Here $P_0$ is the initial pressure gradient.

From a practical viewpoint this problem corresponds to acceleration of a conducting fluid in a magnetohydrodynamic channel with an external circuit connected.

1. We first consider the acceleration of a conducting fluid in a channel with the external secondary circuit open (N = 0).

Letting $I_1(t)$ be a given function of time in Eq. (3), the solution for v(x, t) may be written in the form [2, 3]

$$v(x, t) = \sum_{n=1}^{\infty} \frac{2(\operatorname{ch}(\sqrt{\beta_n}\, 1/2 b) - \operatorname{ch}(\sqrt{\beta_n}\, x))}{\operatorname{ch}\sqrt{\beta_n}\, 1/2 b \left[3 - \frac{M^2}{4}\left(\frac{\beta_n b^2}{M^2} - 1\right)^2\right]} \times$$
$$\times \int_0^t \left[P(t-\tau) - Mr\frac{\nu}{ab}\sqrt{\frac{\sigma}{\eta}} I_1(t-\tau)\right] e^{\alpha_n \tau} d\tau. \quad (8)$$

Here $\alpha_n = \nu(\beta_n - M^2/b^2)$, and the eigenvalues of $\beta_n$ are determined from the following transcendental equation:

$$\beta_n = \frac{M^2}{b^2}\left(1 - \frac{2}{b\sqrt{\beta_n}} \operatorname{th}\frac{b\sqrt{\beta_n}}{2}\right). \quad (9)$$

Analysis of (9) indicates that for $M \gg 1$ there is a unique positive value of $\beta$, which determines, to a high degree of accuracy, the nature of the variation in time of the mean fluid flow velocity.

Substituting (8) into (4), with $M \gg 0$, we obtain

$$L_1 \frac{dI_1}{dt} + (r_1 + r) I_1 =$$
$$= W_+ \int_0^t P(t-\tau) e^{\alpha\tau} d\tau - W_- \int_0^t I_1(t-\tau) e^{\alpha\tau} d\tau,$$
$$W_+ = \frac{2B_0 a b^2 \beta}{M^2[3 - 1/4 M^2(\beta b^2/M^2 - 1)^2]},$$
$$W_- = Mr\frac{\nu}{ab}\sqrt{\frac{\sigma}{\eta}} W_+. \quad (10)$$

Assuming $\tau = t - \theta$, and turning to the new variable $X = I_1 e^{-\alpha t}$, we may write (10), following differentiation with respect to t, in the form

$$\frac{d^2 X}{dt^2} + 2\delta \frac{dX}{dt} + \omega^2 X = mP(t) e^{-\alpha t},$$
$$\delta = 1/2 (r_1 + r + \alpha L_1) L_1^{-1},$$
$$\omega^2 = W_-/L_1, \qquad m = W_+/L_1. \quad (11)$$

The solution of (11), allowing for the initial conditions (7), depending on the sign of the discriminant, can be represented in the form [4] $\lambda$:

a) $\lambda^2 = 4(\delta^2 - \omega^2) > 0$ ,

$$X(t) = \frac{2m}{\lambda} \int_0^t P(\tau)\, e^{-\alpha\tau} e^{\delta(\tau - t)} \operatorname{sh} \frac{\lambda}{2}(t-\tau)\, d\tau ; \qquad (12)$$

b) $\lambda^2 = 4(\omega^2 - \delta^2) > 0$ ,

$$X(t) = \frac{2m}{\lambda} \int_0^t P(\tau)\, e^{-\alpha\tau} e^{\delta(\tau - t)} \sin \frac{\lambda}{2}(t-\tau)\, d\tau ; \qquad (13)$$

c) $\lambda = 0$ ,

$$X(t) = m \int_0^t (t-\tau) P(\tau)\, e^{-\alpha\tau} e^{\delta(\tau - t)}\, d\tau . \qquad (14)$$

Thus, for a given function P(t), depending on the specific conditions from (12)–(14), $I_1(t)$ is determined, and the local fluid flow velocity is calculated from (8).

2. With the external secondary circuit closed ($N \neq 0$), the problem is solved analogously. Introducing the new variables $X_1 = I_1 e^{-\alpha t}$ and $X_2 = I_2 e^{-\alpha t}$, the system of equations (3)–(5) may be written in the form

$$L_1 \frac{d^2X_1}{dt^2} + 2\delta L_1 \frac{dX_1}{dt} + N \frac{d^2X_2}{dt^2} + $$
$$+ \alpha N \frac{dX_2}{dt} + \omega^2 L_1 X_1 = m L_1 P(t)\, e^{-\alpha t},$$
$$N \frac{dX_1}{dt} + L_2 \frac{dX_2}{dt} + \alpha N X_1 + (r_2 + \alpha L_2) X_2 = 0. \qquad (15)$$

Solving this system for $X_1$, we obtain

$$A_1 \frac{d^3X_1}{dt^3} + A_2 \frac{d^2X_1}{dt^2} + A_3 \frac{dX_1}{dt} + A_4 X_1 =$$
$$= m L_1 e^{-\alpha t} P(t) \{A_5 [\ln P(t)]' - \alpha A_5 + A_6\} . \qquad (16)$$

Here

$$A_1 = -(L_1L_2 - N^2),\ A_2 = 2\alpha N^2 -$$
$$-(r_1 + r + \alpha L_1)L_2 - (r_2 + \alpha L_2)\, L_1,$$
$$A_3 = \alpha^2 N^2 - (r_1 + r + \alpha L_1)(r_2 + \alpha L_2) - W_- L_2 ,$$
$$A_4 = -W_- (r_2 + \alpha L_2),\quad A_5 = -L_2,\quad A_6 = -(r_2 + \alpha L_2).$$

Thus, the problem of finding the current $I_1(t)$ reduces to a linear differential equation of the third order with constant coefficients, whose solution for a given function P(t) presents no difficulties.

Figure 2 shows the results of calculating the time variation of the mean flow velocity of the conducting fluid in the magnetohydrodynamic channel under consideration, on sudden application of a constant pressure difference for the case of an open external circuit (curve 1) and a closed circuit (curve 2). The following values were taken as the initial data:

channel dimensions :

$$a = 0.1\text{m},\ b = 0.01\text{m},\ l = 1\text{m} ;$$

working fluid: mercury at T = 20° C,

$$\sigma = 1.046 \cdot 10^6\ 1/\text{ohm} \cdot \text{m} ,$$
$$\eta = 1.55 \cdot 10^{-3}\ \text{N} \cdot \text{sec/m}^2,$$
$$\rho = 13.56 \cdot 10^3\ \text{kg/m}^3,$$
$$r = r_1 = 10^{-5}\ \text{ohm},\quad L_1 = 10^{-5}\ \text{H} ,$$
$$r_2 = 10^{-3}\ \text{ohm},\quad L_2 = 10^{-3}\ \text{H},\ N = 0.5 \cdot 10^{-4}\ \text{H} ,$$
$$M = 100 .$$

The ordinate in the figure shows the mean fluid flow velocity referred to its asymptotic value.

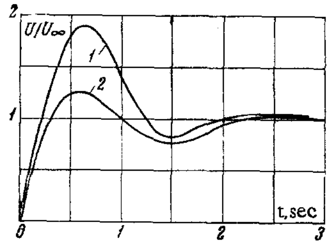


Fig. 2

The investigations that were conducted allow us to conclude that the presence of reactances in the external circuit of a magnetohydrodynamic channel can lead to periodicity in the acceleration of a conducting fluid. In this situation mutual inductance decreases the amplitude of oscillation.

In conclusion we note that this problem can easily be generalized to include capacitance in the external circuit, and nonzero initial conditions.

## REFERENCES


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